

# STRONGLY INTERNAL SETS AND GENERALIZED SMOOTH FUNCTIONS

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**ABSTRACT.** Based on a refinement of the notion of internal sets in Colombeau's theory, so-called strongly internal sets, we introduce the space of generalized smooth functions, a maximal extension of Colombeau generalized functions. Generalized smooth functions as morphisms between sets of generalized points form a sub-category of the category of topological spaces. In particular, they can be composed unrestrictedly.

## 1. INTRODUCTION

Colombeau's nonlinear theory of generalized functions ([7, 8]) is based on viewing generalized functions as equivalence classes of smooth maps, encoding degrees of singularity in terms of asymptotic properties of nets of representatives. It thereby lends itself in a quite straightforward manner to modelling irregular setups in partial differential equations, geometry or applications, in particular in mathematical physics ([9, 20, 16]). Basically, singular objects are modelled as nets of smooth maps and classical operations are lifted to the generalized setting by applying them component-wise to these nets. While successful in applications, this approach lacks strong general existence theorems, comparable to the functional-analytic foundations of distribution theory.

To remedy this situation, the past decade has seen a number of fundamental contributions to the structure theory of algebras of generalized functions (particularly relevant for the purposes of this paper are [1, 2, 3, 4, 5, 11, 12, 13, 15, 28, 29, 30]). The unifying theme of these works is to consider Colombeau generalized functions as set-theoretical functions on suitable spaces of generalized points and then to work directly with these functions.

In the present work we continue these investigations by introducing a generalization of Colombeau-type generalized functions, which we call generalized smooth functions (GSF). This terminology is intended to stress the conceptual analogy between these generalized functions and the theory of standard smooth functions. Generalized smooth functions are set-theoretic maps on sets of generalized points that satisfy the minimal logical conditions necessary to obtain well-defined maps obeying the standard asymptotic estimates of the Colombeau approach. They are the natural extension of Colombeau generalized functions to general domains. At

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the same time, they display optimal set-theoretical properties. In particular, sets of generalized points, together with generalized smooth maps form a subcategory of the category of topological maps.

Our constructions strongly rely on the further development of the concept (itself inspired by nonstandard analysis) of internal sets, see [21]. Just as in the case of classical smooth functions, GSF are locally Lipschitz functions. Therefore, we also study this notion for functions defined on and valued in generalized points.

## 2. BASIC NOTIONS

In this section, we fix some basic notations and terminology from Colombeau's theory. For details we refer to [7, 8, 20, 16]. In the naturals  $\mathbb{N} = \{0, 1, 2, \dots\}$  we include zero. Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and denote by  $I$  the interval  $(0, 1]$ . The (special) Colombeau algebra on  $\Omega$  is defined as the quotient  $\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$  of *moderate nets* over *negligible nets*, where the former is

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and the latter is

$$\mathcal{N}(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)\}.$$

Throughout this paper, every asymptotic relation is for  $\varepsilon \rightarrow 0^+$ . Nets in  $\mathcal{E}_M(\Omega)$  are written as  $(u_\varepsilon)$ , and  $u = [u_\varepsilon]$  denotes the corresponding equivalence class in  $\mathcal{G}(\Omega)$ . For  $(u_\varepsilon) \in \mathcal{N}(\Omega)$  we also write  $(u_\varepsilon) \sim 0$ .  $\Omega \mapsto \mathcal{G}(\Omega)$  is a fine and supple sheaf of differential algebras and there exist sheaf embeddings of the space of Schwartz distributions  $\mathcal{D}'$  into  $\mathcal{G}$  (cf. [16, 23]).

The ring of constants in  $\mathcal{G}$  is denoted by  $\tilde{\mathbb{R}}$  or  $\tilde{\mathbb{C}}$ , respectively, and is called ring of Colombeau generalized numbers (CGN). It is an ordered ring with respect to  $[x_\varepsilon] \leq [y_\varepsilon]$  iff  $\exists [z_\varepsilon] \in \tilde{\mathbb{R}}$  such that  $(z_\varepsilon) \sim 0$  and  $x_\varepsilon \leq y_\varepsilon + z_\varepsilon$  for  $\varepsilon$  sufficiently small. As usual  $x < y$  means  $x \leq y$  and  $x \neq y$ . Even if this order is not total, we still have the possibility to define the infimum  $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$ , and analogously the supremum of two elements. More generally, the space of generalized points in  $\Omega$  is  $\tilde{\Omega} = \Omega_M / \sim$ , where  $\Omega_M = \{(x_\varepsilon) \in \Omega^I \mid \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N})\}$  is called the set of moderate nets and  $(x_\varepsilon) \sim (y_\varepsilon)$  if  $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^m)$  for every  $m \in \mathbb{N}$ . By  $\mathcal{N}$  we will denote the set of all negligible nets of real numbers  $(x_\varepsilon) \in \mathbb{R}^I$ , i.e. such that  $(x_\varepsilon) \sim 0$ . If  $\mathcal{P}(\varepsilon)$  is a property of  $\varepsilon \in I$ , we will also sometimes use the notation  $\forall^0 \varepsilon : \mathcal{P}(\varepsilon)$  to denote  $\exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$ .

The space of compactly supported generalized points  $\tilde{\Omega}_c$  is defined by  $\Omega_c / \sim$ , where  $\Omega_c := \{(x_\varepsilon) \in \Omega^I \mid \exists K \Subset \Omega \exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : x_\varepsilon \in K\}$  and  $\sim$  is the same equivalence relation as in the case of  $\tilde{\Omega}$ . The set of near-standard points of a given set  $A \subseteq \mathbb{R}^n$  is  $A^\bullet := \{x \in \tilde{A} \mid \exists \lim_{\varepsilon \rightarrow 0^+} x_\varepsilon =: x^\circ \in A\}$ . Any Colombeau generalized function (CGF)  $u \in \mathcal{G}(\Omega)$  acts on generalized points from  $\tilde{\Omega}_c$  by  $u(x) := [u_\varepsilon(x_\varepsilon)]$  and is uniquely determined by its point values (in  $\tilde{\mathbb{R}}$ ) on compactly supported generalized points ([16, 19]), but not on standard points. A CGF  $[u_\varepsilon]$  is called compactly-bounded (c-bounded) from  $\Omega$  into  $\Omega'$  if for all  $K \Subset \Omega$  there exists  $K' \Subset \Omega'$  such that  $u_\varepsilon(K) \subseteq K'$  for  $\varepsilon$  small. This type of CGF is closed with respect to composition. Moreover, if  $u \in \mathcal{G}(\Omega)$  is c-bounded from  $\Omega$  into  $\Omega'$  and  $v \in \mathcal{G}(\Omega')$ , then  $[v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}(\Omega)$ .

Our notations for intervals are:  $[a, b] := \{x \in \widetilde{\mathbb{R}} \mid a \leq x \leq b\}$ ,  $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$ . Moreover, for  $x, y \in \widetilde{\mathbb{R}}^n$  we will write  $x \approx y$  if  $x - y$  is infinitesimal, i.e. if  $|x - y| \leq r$  for all  $r \in \mathbb{R}_{>0}$ .

Topological methods in Colombeau's theory are usually based on the so-called sharp topology ([6, 24, 25, 3, 4, 18, 13]), which is the topology generated by balls  $B_\rho(x) = \{y \in \widetilde{\mathbb{R}}^n \mid |y - x| < \rho\}$ , where  $|\cdot|$  is the natural extension of the Euclidean norm to  $\widetilde{\mathbb{R}}^n$ , i.e.  $||x_\varepsilon|| := ||x_\varepsilon||$ , and  $\rho \in \widetilde{\mathbb{R}}_{>0}$  is positive invertible ([1, 2, 15]). Henceforth, we will also use the notation  $B_\rho^e(x) = \{y \in \mathbb{R}^n \mid |y - x| < \rho\}$  for Euclidean balls and  $\widetilde{\mathbb{R}}^* := \{x \in \widetilde{\mathbb{R}} \mid x \text{ is invertible}\}$ . The sharp topology can also be defined by an ultrametric: Define a pseudovaluation on  $\widetilde{\mathbb{R}}$  by

$$\begin{aligned} v : \mathcal{E}_M &\longrightarrow (-\infty, \infty] \\ v((u_\varepsilon)) &:= \sup\{b \in \mathbb{R} \mid |u_\varepsilon| = O(\varepsilon^b)\}. \end{aligned}$$

Then  $v$  is well-defined since  $v((u_\varepsilon) + (n_\varepsilon)) = v((u_\varepsilon))$  for all  $(n_\varepsilon) \in \mathcal{N}$ ;  $v(u) = \infty$  if and only if  $u = 0$ ,  $v(u \cdot w) \geq v(u) + v(w)$ ,  $v(u + w) \geq v(u) \wedge v(w)$ , and  $v(u - w) = v(w - u)$ . Letting  $|\cdot|_e : \widetilde{\mathbb{R}} \rightarrow [0, \infty)$ ,  $|u|_e := \exp(-v(u))$  it follows that  $|u + v|_e \leq \max(|u|_e, |v|_e)$ , as well as  $|uv|_e \leq |u|_e |v|_e$ . This induces the translation invariant complete ultrametric

$$\begin{aligned} d_s : \widetilde{\mathbb{R}} \times \widetilde{\mathbb{R}} &\longrightarrow \mathbb{R}_+ \\ d_s(u, v) &:= |u - v|_e \end{aligned}$$

on  $\widetilde{\mathbb{R}}$ , which in turn generates the sharp topology on  $\widetilde{\mathbb{R}}$ . We will call sharply open any open set in the sharp topology.

Moreover, Garetto in [11, 12] extended the above construction to arbitrary locally convex spaces by functorially assigning a space of CGF  $\mathcal{G}_E$  to any given locally convex space  $E$ . The seminorms of  $E$  can then be used to define pseudovaluations which in turn induce a generalized locally convex topology on the  $\widetilde{\mathbb{C}}$ -module  $\mathcal{G}_E$ , again called sharp topology.

Given  $S \subseteq I$ , by  $e_S$  we will denote the equivalence class in  $\widetilde{\mathbb{R}}$  of the characteristic function of  $S$ . The  $e_S$  are idempotents, and satisfy  $e_S + e_{S^c} = 1$  and  $e_S \neq 0$  if and only if  $0 \in \overline{S}$ . They play a central role in the algebraic theory of Colombeau generalized numbers (cf. [3, 29]).

Finally, we recall that if  $(A_\varepsilon)$  is a net of subsets of  $\mathbb{R}^n$ , then the internal set generated by  $(A_\varepsilon)$  is defined as

$$[A_\varepsilon] = \left\{ [x_\varepsilon] \in \widetilde{\mathbb{R}}^n \mid x_\varepsilon \in A_\varepsilon \text{ for } \varepsilon \text{ small} \right\}. \quad (2.1)$$

This type of sets have been introduced in Colombeau theory in [21, 30] to deal with domains of generalized functions and to study topological properties of sets of generalized points.

### 3. STRONGLY INTERNAL SETS GENERATED BY A TOPOLOGY

We start by defining a family of topologies on  $\widetilde{\mathbb{R}}^n$  depending on a set of positive and invertible generalized numbers. Recall that for any  $r \in \widetilde{\mathbb{R}}$ ,  $r > 0$ ,  $B_r(x)$  denotes the open ball with respect to the generalized Euclidean norm in  $\widetilde{\mathbb{R}}^n$ .

**Definition 1.** We say that  $\mathcal{I}$  is a *set of radii* if

- (i)  $\mathcal{I} \subseteq \widetilde{\mathbb{R}}_{>0}^*$  is a non-empty subset of positive invertible generalized numbers.

- (ii) For all  $r, s \in \mathcal{I}$  the infimum  $r \wedge s \in \mathcal{I}$ .
- (iii)  $k \cdot r \in \mathcal{I}$  for all  $r \in \mathcal{I}$  and all  $k \in \mathbb{R}_{>0}$ .

Let  $\mathcal{I}$  be a set of radii, then the family of subsets

$$\mathcal{U}_{\mathcal{I}}(x) := \{U \subseteq \tilde{\mathbb{R}}^n \mid \exists r \in \mathcal{I} : B_r(x) \subseteq U\} \quad (x \in \tilde{\mathbb{R}}^n)$$

is called the *neighborhood system induced by  $\mathcal{I}$* .

The legitimacy of this name is demonstrated by the following result, whose proof follows from the corresponding definitions.

**Theorem 2.** *If  $\mathcal{I}$  is a set of radii, then the family  $\mathcal{U}_{\mathcal{I}}$  is a non empty neighborhood system on  $\tilde{\mathbb{R}}^n$ . The topology  $\tau_{\mathcal{I}}$  induced by this neighborhood system is called the topology on  $\tilde{\mathbb{R}}^n$  induced by the set of radii  $\mathcal{I}$ .*

**Example 3.**

- (i) If  $\mathcal{I} = \tilde{\mathbb{R}}_{>0}^*$  then  $\tau_{\mathcal{I}}$  is the sharp topology. Among the balls  $B_r(x)$  in this topology we can also have cases where both  $r \in \mathcal{I}$  and  $x \in \tilde{\mathbb{R}}^n$  are not near standard.
- (ii) If  $\mathcal{I} = \mathbb{R}_{>0}$  then  $\tau_{\mathcal{I}}$  is called the Fermat topology on  $\tilde{\mathbb{R}}^n$ . Among the balls  $B_r(x)$  in this topology we can only have cases where the radius is a standard positive real number. For this reason, open sets in this topology are also called *large open sets*. Let us note that if  $S \subseteq \mathbb{R}^\bullet$ , i.e. if  $S$  consists of near-standard points only (see Section 2), then the trace of the Fermat topology on  $S$  is induced by the Fermat pseudometric  $d_F(x, y) = |x^\circ - y^\circ|$ . In fact, if  $B_r^F(x) = \{y \in \mathbb{R}^\bullet \mid |y^\circ - x^\circ| < r\}$  are the balls in this metric, then  $B_{r/2}(x) \cap S \subseteq B_r^F(x) \cap S \subseteq B_r(x) \cap S$ . This justifies the name Fermat topology (introduced in [15]) for the topology of large open sets.
- (iii) Let  $a \in \mathbb{R}_{>0}$ , and set  $\mathcal{I}_a := \{[r \cdot \varepsilon^b] \in \tilde{\mathbb{R}} \mid r \in \mathbb{R}_{>0}, 0 < b < a\}$ , then  $\mathcal{I}_a$  is a set of radii that generates a topology (on  $\tilde{\mathbb{R}}^n$ ) strictly coarser than the sharp and strictly finer than the Fermat ones. Open sets defined by  $\mathcal{I}_a$  cannot contain neighborhoods of radius smaller than  $[\varepsilon^a]$ .
- (iv) Let  $H \subseteq \tilde{\mathbb{R}}_{>0}^*$  be a non empty set of positive and invertible CGN, then  $\mathcal{J}_H := \{\bigwedge_{i=1}^n r_i \cdot h_i \mid n \in \mathbb{N}_{>0}, r_i \in \mathbb{R}_{>0}, h_i \in H\}$  is the smallest set of radii containing  $H$ . In particular, if  $H = \{h\}$  and  $h \approx 0$  then  $\mathcal{J}_{\{h\}} = \{r \cdot h \mid r \in \mathbb{R}_{>0}\}$  generates a topology strictly finer than the Fermat one and strictly coarser than the sharp one. On the contrary, if  $h$  is infinite, i.e.  $|h| > s$  for all  $s \in \mathbb{R}_{>0}$ , then it generates a topology strictly coarser than the Fermat one. Finally,  $\mathcal{J}_{\{[\varepsilon^a]\}}$  generates a topology strictly finer than the topology generated by the set of radii  $\mathcal{I}_a$  described in (iii).

In the present work, we will only develop examples (i) and (ii).

Any topology on  $\tilde{\mathbb{R}}^n$  can be used to introduce an equivalence relation on  $\tilde{\mathbb{R}}^n$  which permits to define a corresponding class of strongly internal sets:

**Definition 4.** Let  $\tau$  be a topology on  $\tilde{\mathbb{R}}^n$ , and  $x, y \in \tilde{\mathbb{R}}^n$ , then we say that  $x, y$  are identified by  $\tau$ , and we write  $x \asymp_\tau y$  if for all  $\tau$ -open sets  $U \in \tau$

$$x \in U \iff y \in U.$$

Clearly,  $\asymp_\tau$  is an equivalence relation on  $\tilde{\mathbb{R}}^n$ .

**Example 5.**

- (i) If  $\tau$  is the sharp topology, then  $x \asymp_\tau y$  if and only if  $x = y$ .
- (ii) If  $\tau$  is the Fermat topology, then  $x \asymp_\tau y$  if and only if  $x \approx y$ .

The following notion concerns membership for  $\varepsilon$ -dependent objects; it assures that the class of nets we will consider is always closed under choosing different representatives with respect to  $\asymp_\tau$ .

**Definition 6.** Let  $(A_\varepsilon)$  be a net of subsets of  $\mathbb{R}^n$ . Moreover, let  $(x_\varepsilon)$  be a net of points in  $\mathbb{R}_M^n$ , then we say that  $(x_\varepsilon)$   $\tau$ -strongly belongs to  $(A_\varepsilon)$  and we write

$$x_\varepsilon \in_\tau A_\varepsilon$$

if

- (i)  $x_\varepsilon \in A_\varepsilon$  for  $\varepsilon$  sufficiently small;
- (ii) If  $[x'_\varepsilon] \asymp_\tau [x_\varepsilon]$ , then also  $x'_\varepsilon \in A_\varepsilon$  for  $\varepsilon$  sufficiently small.

Therefore, we can consider the set

$$\langle A_\varepsilon \rangle_\tau := \{[x_\varepsilon] \in \widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\tau A_\varepsilon\} \quad (3.1)$$

which, generally speaking, is a subset of the corresponding internal set

$$[A_\varepsilon] = \left\{ [x_\varepsilon] \in \widetilde{\mathbb{R}}^n \mid x_\varepsilon \in A_\varepsilon \text{ for } \varepsilon \text{ small} \right\}, \quad (3.2)$$

as defined in [21, 30], because of our definition of strong membership. Subsets of  $\widetilde{\mathbb{R}}^n$  of the form (3.1) will be called  $\tau$ -strongly internal. In particular we simply use the name *strongly internal set* for the case where  $\tau$  is the sharp topology and *large internal set* for the case where  $\tau$  is the Fermat topology. In the first one, we use the notations  $\in_\varepsilon$  and  $\langle A_\varepsilon \rangle$ ; in the second one we use  $\in_F$  and  $\langle A_\varepsilon \rangle_F$ .

*Remark 7.*

- (i)  $\widetilde{\mathbb{R}} = \langle (-e^{\frac{1}{\varepsilon}}, e^{\frac{1}{\varepsilon}}) \rangle$  is strongly internal.
- (ii) If  $\Omega \subseteq \mathbb{R}^n$  is any open set, then  $\widetilde{\Omega}_c$  is a large open set. In fact, if  $x \in \widetilde{\Omega}_c$ , we have  $x_\varepsilon \in K \Subset \Omega$  for  $\varepsilon$  small. Since  $K$  is compact,  $d(K, \mathbb{R}^n \setminus \Omega) > 0$ . Taking  $r \in \mathbb{R}_{>0}$  strictly less than this distance, any  $[y_\varepsilon] \in B_r(x)$  is compactly supported as well.
- (iii) It is easy to prove that  $\langle A_\varepsilon \cap B_\varepsilon \rangle = \langle A_\varepsilon \rangle \cap \langle B_\varepsilon \rangle$ , whereas the corresponding property for internal sets is false in general.
- (iv) Let  $\mathcal{P}(-)$  be a property of generalized points in  $\widetilde{\mathbb{R}}^n$  (i.e.  $\mathcal{P}(-)$  has to be thought of as a syntactical object), and set  $\ulcorner \mathcal{P} \urcorner := \{x \in \widetilde{\mathbb{R}}^n \mid \mathcal{P}(x)\}$  (i.e.  $\ulcorner \mathcal{P} \urcorner$  is the set-theoretical interpretation of  $\mathcal{P}$ ). For  $x, y \in \widetilde{\mathbb{R}}^n$ , we have that  $x \asymp_\tau y$  if and only if for each property  $\mathcal{P}$ , if  $\ulcorner \mathcal{P} \urcorner \in \tau$  then  $\mathcal{P}(x)$  holds if and only if  $\mathcal{P}(y)$  holds. We can say that  $x$  and  $y$  are identified by  $\tau$  if and only if these generalized points have the same properties  $\mathcal{P}(-)$  which can be interpreted as open sets in the topology  $\tau$ , i.e. such that the set-theoretical interpretation  $\ulcorner \mathcal{P} \urcorner$  of the property  $\mathcal{P}$  is an open set in  $\tau$ . Moreover, if  $\mathcal{P}$  is one of these properties and  $\mathcal{P}(x)$  holds, then we can say it is a  $\tau$ -stable property, i.e. also  $\mathcal{P}(y)$  holds for  $y$  sufficiently near to  $x$  with respect to  $\tau$ , i.e. if  $y \asymp_\tau x$ .

The following result provides a certain geometrical intuition about this notion of  $\tau$ -strong membership and justifies its name. It also underscores the differences with internal sets as studied in [21, 30].

**Theorem 8.** Let  $(A_\varepsilon)$  be a net of subsets of  $\mathbb{R}^n$  indexed for  $\varepsilon \in I$ , and let  $(x_\varepsilon) \in \mathbb{R}_M^n$ . Then the following properties hold:

- (i)  $x_\varepsilon \in_\varepsilon A_\varepsilon$  if and only if there exists some  $q \in \mathbb{R}_{>0}$  such that  $d(x_\varepsilon, A_\varepsilon^c) > \varepsilon^q$  for  $\varepsilon$  sufficiently small, where  $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$ . Therefore,  $x_\varepsilon \in_\varepsilon A_\varepsilon$  if and only if  $[d(x_\varepsilon, A_\varepsilon^c)]$  is invertible in  $\widetilde{\mathbb{R}}$ .
- (ii)  $x_\varepsilon \in_F A_\varepsilon$  if and only if there exists some  $r \in \mathbb{R}_{>0}$  such that  $d(x_\varepsilon, A_\varepsilon^c) > r$  for  $\varepsilon$  sufficiently small.

*Proof.* We proceed for the sharp topology, since the case of the Fermat one can be treated analogously. Let  $x_\varepsilon \in_\varepsilon A_\varepsilon$  and suppose to the contrary that there exists a sequence  $\varepsilon_k \searrow 0$  such that  $d(x_{\varepsilon_k}, A_{\varepsilon_k}^c) \leq \varepsilon_k^k$  for all  $k \in \mathbb{N}$ . For each  $k$ , pick some  $x'_k \in A_{\varepsilon_k}^c$  with  $|x'_k - x_{\varepsilon_k}| < 2\varepsilon_k^k$  and choose  $(x'_\varepsilon) \sim (x_\varepsilon)$  such that  $x'_{\varepsilon_k} = x'_k$  for all  $k$ . Then  $x'_{\varepsilon_k} \notin A_{\varepsilon_k}$  for all  $k$ , contradicting  $x_\varepsilon \in_\varepsilon A_\varepsilon$ . Conversely, let  $d(x_\varepsilon, A_\varepsilon^c) > \varepsilon^q$  for  $\varepsilon$  small. Then in particular,  $x_\varepsilon \in A_\varepsilon$ . Also, if  $(x'_\varepsilon) \sim (x_\varepsilon)$  then  $d(x'_\varepsilon, A_\varepsilon^c) > (1/2)\varepsilon^q$  for  $\varepsilon$  small, so  $x'_\varepsilon \in A_\varepsilon$ . Thus,  $x_\varepsilon \in_\varepsilon A_\varepsilon$ .  $\square$

Hence for  $x = [x_\varepsilon] \in \widetilde{\mathbb{R}}^n$ ,  $x_\varepsilon \in_\varepsilon A_\varepsilon$  if and only if  $[x_\varepsilon]$  is in the interior of  $\langle A_\varepsilon \rangle$  with respect to the sharp topology.

**Corollary 9.**  $\langle A_\varepsilon \rangle = \langle \overset{\circ}{A}_\varepsilon \rangle$  is open in the sharp topology.

Note that even in the simplest case of a constant net  $\Omega = \Omega_\varepsilon$ , the corresponding strongly internal set  $\langle \Omega \rangle$  is contained in  $\widetilde{\Omega}$ , but equality in general does not hold (see (ii) in Example 10).

**Example 10.**

- (i) Let  $(a_\varepsilon), (b_\varepsilon) \in \mathbb{R}_M$ , with  $a_\varepsilon < b_\varepsilon$ , then from Prop. 8 it is easy to prove that  $\langle [a_\varepsilon, b_\varepsilon] \rangle = \langle (a_\varepsilon, b_\varepsilon) \rangle = \{x \in (a, b) \mid x - a, b - x \in \widetilde{\mathbb{R}}_{>0}^*\}$ , where we recall that  $\widetilde{\mathbb{R}}^* := \{x \in \mathbb{R} \mid x \text{ is invertible}\}$ .
- (ii) We always have that  $\langle A_\varepsilon \rangle \subseteq \text{int}_s([A_\varepsilon])$ , where  $[A_\varepsilon]$  is the internal set generated by the net  $(A_\varepsilon)_\varepsilon$  in the sense of [21, 30] (see (3.2)) and  $\text{int}_s(B)$  is the interior of  $B \subseteq \widetilde{\mathbb{R}}^n$  in the sharp topology. Indeed,  $\langle A_\varepsilon \rangle \subseteq [A_\varepsilon]$  by definition, and  $\langle A_\varepsilon \rangle$  is open in the sharp topology by Corollary 9. However, the reverse inclusion is false: let  $A_\varepsilon = \mathbb{R} \setminus \{0\}$ . Then  $\langle A_\varepsilon \rangle = \widetilde{\mathbb{R}}^* \subsetneq \text{int}_s([A_\varepsilon]) = \text{int}_s(\mathbb{R}) = \mathbb{R}$ .

We close this section with the following result, which provides a certain intuition on the net of open sets  $\Omega_\varepsilon \subseteq \mathbb{R}^n$  that generates the strongly internal set  $\langle \Omega_\varepsilon \rangle$ . We recall that a net  $(B_\varepsilon)$  of subsets of  $\mathbb{R}^n$  is called *sharply bounded* if there exists  $N \in \mathbb{R}_{>0}$  such that

$$\forall^0 \varepsilon \forall a \in B_\varepsilon : |a| \leq \varepsilon^{-N}.$$

**Theorem 11.** Let  $(\Omega_\varepsilon)$  be a net of subsets in  $\mathbb{R}^n$  for all  $\varepsilon$ , and  $(B_\varepsilon)$  a sharply bounded net such that  $[B_\varepsilon] \subseteq \langle \Omega_\varepsilon \rangle$ , then

$$\forall^0 \varepsilon : B_\varepsilon \subseteq \Omega_\varepsilon.$$

*Proof.* By contradiction assume that we can find sequences  $(\varepsilon_k)_k$  and  $(x_k)_k$  such that  $\varepsilon_k \downarrow 0$  and  $x_k \in B_{\varepsilon_k} \setminus \Omega_{\varepsilon_k}$ . Defining  $x_\varepsilon := x_k$  for  $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$ , and  $x_\varepsilon \in B_\varepsilon$  otherwise, we have that  $x := [x_\varepsilon]$  is moderate since  $(B_\varepsilon)$  is sharply bounded. Hence  $x \in [B_\varepsilon]$ , but  $x_{\varepsilon_k} \notin \Omega_{\varepsilon_k}$  by construction, hence  $x \notin \langle \Omega_\varepsilon \rangle$  by Def. 6, which is impossible because  $[B_\varepsilon] \subseteq \langle \Omega_\varepsilon \rangle$ .  $\square$

**Example 12.** Let  $\Omega \subseteq \mathbb{R}^n$  be open and bounded. Then  $\widetilde{\Omega}_c$  is not strongly internal. Indeed, suppose that  $\widetilde{\Omega}_c = \langle \Omega_\varepsilon \rangle$ . Let  $(K_n)_n$  be a compact exhaustion of  $\Omega$ . Then by the previous theorem, there exist  $\varepsilon_n$  such that  $K_n \subseteq \Omega_\varepsilon$ , for each  $\varepsilon \leq \varepsilon_n$ .

W.l.o.g.,  $(\varepsilon_n)_n$  decreasingly tends to 0 and  $\varepsilon_n \leq d_n := d(K_{n-1}, K_n^c)$ . Choose  $x_\varepsilon \in K_{n-1} \setminus \widetilde{K_{n-2}}$  for each  $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$ . As  $\Omega$  is bounded,  $(x_\varepsilon)_\varepsilon$  is moderate. Then  $[x_\varepsilon] \notin \widetilde{K_n}$ , for each  $n \in \mathbb{N}$ . On the other hand,  $[x_\varepsilon] \in \langle \Omega_\varepsilon \rangle$ , since  $d(x_\varepsilon, \Omega_\varepsilon^c) \geq \varepsilon$ . For, if  $|y_\varepsilon - x_\varepsilon| \leq \varepsilon$  and  $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$ , then  $|y_\varepsilon - x_\varepsilon| \leq d_n$  and  $x_\varepsilon \in K_{n-1}$ . Hence  $y_\varepsilon \in K_n \subseteq \Omega_\varepsilon$ .

The following theorem sheds some light on the relationship between internal sets and strongly internal sets, implying e.g. that they generate the same  $\sigma$ -algebra:

**Theorem 13.** *Let  $A_\varepsilon \subseteq \mathbb{R}^n$ . Then we have:*

- (i)  $\langle A_\varepsilon \rangle = \bigcup_{m \in \mathbb{N}} [A_{m,\varepsilon}]$ , where  $A_{m,\varepsilon} = \{x \in \mathbb{R}^n : d(x, A_\varepsilon^c) \geq \varepsilon^m\}$ .
- (ii)  $[A_\varepsilon] = \bigcap_{m \in \mathbb{N}} \langle A_{m,\varepsilon} \rangle$ , where  $A_{m,\varepsilon} = \{x \in \mathbb{R}^n : d(x, A_\varepsilon) < \varepsilon^m\}$ .

*Proof.* (i) Let  $[x_\varepsilon] \in \widetilde{\mathbb{R}^n}$ . By Theorem 8,  $[x_\varepsilon] \in \langle A_\varepsilon \rangle$  iff  $d(x_\varepsilon, A_\varepsilon^c) \geq \varepsilon^m$  for small  $\varepsilon$ , for some  $m \in \mathbb{N}$ .

(ii) Let  $[x_\varepsilon] \in \widetilde{\mathbb{R}^n}$ . By [21, Prop. 2.1],  $[x_\varepsilon] \in [A_\varepsilon]$  iff  $d(x_\varepsilon, A_\varepsilon) \sim 0$  iff  $d(x_\varepsilon, A_\varepsilon) < \varepsilon^m$  for each  $m \in \mathbb{N}$  (and for each representative  $(x_\varepsilon)$ ).  $\square$

**Theorem 14.** *Let  $(A_\varepsilon)$  be sharply bounded. Then*

$$\langle A_\varepsilon \rangle \subseteq \langle B_\varepsilon \rangle \iff \sup_{x \in B_\varepsilon^c} d(x, A_\varepsilon^c) \sim 0.$$

*Proof.*  $\Rightarrow$ : by the previous theorems, we have  $[A_{m,\varepsilon}] \subseteq \langle B_\varepsilon \rangle$ , and thus  $A_{m,\varepsilon} \subseteq B_\varepsilon$ , i.e.,  $B_\varepsilon^c \subseteq A_{m,\varepsilon}^c$  for small  $\varepsilon$ , for each  $m$ , where  $A_{m,\varepsilon} = \{x \in \mathbb{R}^n : d(x, A_\varepsilon^c) \geq \varepsilon^m\}$ .  $\Leftarrow$ : if  $x \notin \langle B_\varepsilon \rangle$ , then there exists a representative  $(x_\varepsilon)$  and  $\varepsilon_n \rightarrow 0$  such that  $x_{\varepsilon_n} \in B_{\varepsilon_n}^c$  for each  $n$ . It is given that there exist  $x'_{\varepsilon_n} \in A_{\varepsilon_n}^c$  such that  $|x_{\varepsilon_n} - x'_{\varepsilon_n}| \leq \nu_{\varepsilon_n}$  with  $\nu_\varepsilon \sim 0$ . Let  $x'_\varepsilon := x_\varepsilon$  if not yet defined. Then  $x = [x'_\varepsilon] \notin \langle A_\varepsilon \rangle$ .  $\square$

**Corollary 15.** *If  $(A_\varepsilon)$  and  $(B_\varepsilon)$  are sharply bounded nets, then  $\langle A_\varepsilon \rangle = \langle B_\varepsilon \rangle$  if and only if the Hausdorff distance  $d_H(A_\varepsilon^c, B_\varepsilon^c) \sim 0$ .*

**Definition 16.**  $A \subseteq \widetilde{\mathbb{R}^n}$  is convex if for each  $x, y \in A$  and  $t \in \widetilde{[0, 1]}$ ,  $tx + (1-t)y \in A$ .

**Lemma 17.** *If  $A \subseteq \widetilde{\mathbb{R}^n}$  is convex, internal and sharply bounded, then  $A$  has a representative consisting of convex sets.*

*Proof.* By [21],  $A = [A_\varepsilon]$  for some sharply bounded net  $(A_\varepsilon)_\varepsilon$ . We show that  $A = [\text{conv}(A_\varepsilon)]$ , where  $\text{conv}(X)$  denotes the convex closure of  $X$ . Let  $x \in [\text{conv}(A_\varepsilon)]$ . Then  $x_\varepsilon \in \text{conv}(A_\varepsilon)$  for sufficiently small  $\varepsilon$  and for some representative  $(x_\varepsilon)$  of  $x$ . By Carathéodory's theorem in convex geometry, there exist  $a_{0,\varepsilon}, \dots, a_{n,\varepsilon} \in A_\varepsilon$  such that  $x_\varepsilon \in \text{conv}\{a_{0,\varepsilon}, \dots, a_{n,\varepsilon}\}$ , i.e., there exist  $\lambda_{0,\varepsilon}, \dots, \lambda_{n,\varepsilon} \in [0, 1]_\mathbb{R}$  such that  $x_\varepsilon = \sum_{j=0}^n \lambda_{j,\varepsilon} a_{j,\varepsilon}$ . Since  $(A_\varepsilon)_\varepsilon$  is sharply bounded,  $[a_{j,\varepsilon}] = a_j$  for some  $a_j \in A \subseteq \widetilde{\mathbb{R}^n}$ . Hence  $x = \sum_{j=0}^n \lambda_j a_j \in A$  (with  $\lambda_j = [\lambda_{j,\varepsilon}] \in \widetilde{[0, 1]}$  and  $\sum_{j=0}^n \lambda_j = 1$ ), since  $A \subseteq \widetilde{\mathbb{R}^n}$  is assumed to be convex.  $\square$

#### 4. LOCALLY LIPSCHITZ FUNCTIONS

**Definition 18.** Let  $U \subseteq \widetilde{\mathbb{R}^n}$ . Then  $f : U \rightarrow \widetilde{\mathbb{R}^m}$  is called *Lipschitz* if there exists some  $L \in \widetilde{\mathbb{R}}$  such that  $|f(x) - f(y)| \leq L|x - y|$  for all  $x, y \in U$ , where  $|\cdot|$  is the natural extension of the Euclidean norm to generalized points, i.e.  $|[x_\varepsilon]| := |[x_\varepsilon]|$ . The function  $f$  is called *locally Lipschitz* with respect to some topology  $\tau$  on  $U$  if every  $x \in U$  possesses a  $\tau$ -neighborhood in which  $f$  is Lipschitz in this sense.



It is immediate from the definition that a map  $f : U \rightarrow \widetilde{\mathbb{R}}^m$  is Lipschitz if and only if  $\exists N \in \mathbb{N} : |f(x) - f(y)| \leq [\varepsilon^{-N}]|x - y|$  for all  $x, y \in U$ . Moreover, any locally Lipschitz function in the sharp topology (in particular any locally Lipschitz function in the Fermat topology) is also continuous in the sharp topology.

**Example 19.** As we will see in the next section, any map  $u : \widetilde{\Omega}_c \rightarrow \widetilde{\mathbb{R}}$  generated by a CGF or any map  $u : \widetilde{\mathbb{R}}^n \rightarrow \widetilde{\mathbb{R}}$  generated by a tempered generalized function is locally Lipschitz for the Fermat topology.

The following result shows, in particular, that the composition of locally Lipschitz maps in the sharp topology is again locally Lipschitz, and gives sufficient conditions for the corresponding property in the Fermat topology.

**Lemma 20.** *Let  $A \subseteq \widetilde{\mathbb{R}}^a$ ,  $B \subseteq \widetilde{\mathbb{R}}^b$ ,  $C \subseteq \widetilde{\mathbb{R}}^c$ ,  $D \subseteq \widetilde{\mathbb{R}}^d$  and  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be locally Lipschitz maps in the topology  $\tau$ . Then, if  $\tau$  is the Fermat topology, we have*

- (i) *If  $f$  is locally Lipschitz with respect to finite Lipschitz constants, then it is also continuous in the Fermat topology.*
- (ii) *If  $B = C$  and  $f$  is continuous in the Fermat topology, then  $g \circ f$  is locally Lipschitz.*
- (iii) *If  $B = C$  and  $g$  is Lipschitz, then  $g \circ f$  is locally Lipschitz.*

Whereas if  $\tau$  is the sharp topology and  $B = C$ , then  $g \circ f$  is locally Lipschitz in the same topology.

*Proof.* (i): one easily sees that  $f$  is continuous for the Fermat topology iff

$$\forall x \in A \forall \varepsilon \in \mathbb{R}_{>0} \exists \delta \in \mathbb{R}_{>0} \forall y \in A : (|x - y| \leq \delta \implies |f(x) - f(y)| \leq \varepsilon),$$

which is clearly satisfied if  $f$  satisfies the given conditions of (i).

The proofs of the other parts are formally equal to the standard ones in metric spaces.  $\square$

**Remark 21.** We emphasize that our notion of Lipschitz map differs from the classical definition in a metric space, e.g. with respect to the sharp metrics on  $\widetilde{\mathbb{R}}^n$ ,  $\widetilde{\mathbb{R}}^m$ , because both the Lipschitz constant  $L$  and the generalized metric  $|x - y|$  assume values in  $\widetilde{\mathbb{R}}$ . On the other hand, it is the natural generalization of the classical notion to the non-Archimedean ring  $\widetilde{\mathbb{R}}$ . In fact, if  $U \subseteq \mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}^m$  is Lipschitz in the usual sense, then it is also Lipschitz in the sense of Def. 18. Moreover, if this  $f : U \rightarrow \mathbb{R}^m$  is locally Lipschitz in the usual sense with respect to the Euclidean topology, then viewing  $f$  as a CGF (i.e. through the embedding  $\mathcal{C}^0(U, \mathbb{R}^m) \subseteq \mathcal{D}'(U)^m \subseteq \mathcal{G}(U)^m$ ), it is easy to prove that the induced map  $f : \widetilde{U}_c \rightarrow \widetilde{\mathbb{R}}^m$  (which extends the original  $f$ ) is locally Lipschitz with respect to the Fermat topology with finite Lipschitz constant.

While clearly on  $\mathbb{R}$  (as in any metric space) from a local Lipschitz condition it is possible to obtain a global one on compact sets, this is not directly translatable into  $\widetilde{\mathbb{R}}$  with the above concept of locally Lipschitz maps. In fact, this property already fails on finite sets. E.g., let  $U = \{0, e_S\} \subseteq \widetilde{\mathbb{R}}$ , where  $e_S \neq 0$  is a zero divisor, and let  $f(0) := 0$  and  $f(e_S) := 1$ . Then  $f$  is locally Lipschitz for the Fermat topology, but not globally Lipschitz, since  $1 = |f(e_S) - f(0)| \leq C|e_S - 0| = C e_S$  does not hold for any  $C \in \widetilde{\mathbb{R}}$ . We still have the following:



**Definition 22.** Let  $U \subseteq \widetilde{\mathbb{R}}^n$ . We call  $f: U \rightarrow \widetilde{\mathbb{R}}^m$  *pointwise Lipschitz* if for each  $x, y \in U$ , there exists some  $C \in \widetilde{\mathbb{R}}$  such that  $|f(y) - f(x)| \leq C|y - x|$ .

We call  $f$  *strongly locally Lipschitz* w.r.t. the topology  $\tau$  if every  $x, y \in U$  possess  $\tau$ -neighbourhoods  $V_x$  and  $V_y$  respectively such that  $f$  is Lipschitz on  $V_x \cup V_y$ .

**Theorem 23.** Let  $\tau$  be either the sharp topology or the Fermat topology. Let  $K \subseteq \widetilde{\mathbb{R}}^n$  be  $\tau$ -compact.

- (i) If  $f: K \rightarrow \widetilde{\mathbb{R}}^m$  is  $\tau$ -strongly locally Lipschitz on  $K$ , then  $f$  is globally Lipschitz on  $K$ .
- (ii) Let  $f: K \rightarrow \widetilde{\mathbb{R}}^m$  be a  $\tau$ -locally Lipschitz and pointwise Lipschitz map. Let for each  $x, y \in K$  with  $x \not\approx_\tau y$  necessarily  $|x - y| \geq [\varepsilon^m]$  for some  $m \in \mathbb{N}$  (if  $\tau$  is the sharp topology), resp.  $|x - y| \geq r$  for some  $r \in \mathbb{R}_{>0}$  (if  $\tau$  is the Fermat topology). Then  $f$  is globally Lipschitz on  $K$ .

*Proof.* (i) For each  $n \in \mathbb{N}$ , call  $A_n$  the  $\tau$ -interior of the set  $\{(x, y) \in K \times K : |f(y) - f(x)| \leq [\varepsilon^{-n}]|y - x|\}$ . Since  $f$  is strongly locally Lipschitz on  $K$ , every  $(x, y) \in K \times K$  belongs to  $A_n$  for some  $n \in \mathbb{N}$ . In fact,  $(A_n)_{n \in \mathbb{N}}$  is a  $\tau$ -open cover of  $K \times K$ . As  $K \times K$  is  $\tau$ -compact, it follows that  $K \times K \subseteq A_N$  for some  $N \in \mathbb{N}$ . Hence  $f$  is Lipschitz on  $K$ .

(ii): by (i), we only have to show that  $f$  is  $\tau$ -strongly locally Lipschitz on  $K$ . Thus consider any  $x, y \in K$ . Choose a  $\tau$ -neighbourhood  $V_x$  of  $x$  (resp.  $V_y$  of  $y$ ) on which  $f$  is Lipschitz. If  $x = y$ , then  $V_x$  is also a  $\tau$ -neighbourhood of  $y$ , and thus  $f$  is trivially Lipschitz on  $V_x = V_x \cup V_y$ . Otherwise,  $|x - y| \geq [\varepsilon^m]$  for some  $m \in \mathbb{N}$  by assumption. By shrinking  $V_x$  and  $V_y$ , we may assume that  $V_x \subseteq B_{[\varepsilon^m]/3}(x)$  and  $V_y \subseteq B_{[\varepsilon^m]/3}(y)$ . Then there exists  $N \in \mathbb{N}$  such that for any  $\xi \in V_x$  and  $\eta \in V_y$

$$\begin{aligned} |f(\eta) - f(\xi)| &\leq |f(\eta) - f(y)| + |f(y) - f(x)| + |f(x) - f(\xi)| \\ &\leq [\varepsilon^{-N}]|\eta - y| + [\varepsilon^{-N}]|y - x| + [\varepsilon^{-N}]|x - \xi| \end{aligned}$$

since  $f$  is Lipschitz on  $V_x$  and on  $V_y$  and pointwise Lipschitz. Now  $|x - \xi| \leq [\varepsilon^m]/3 \leq |x - y|/3$  and  $|y - \eta| \leq |x - y|/3$ . Hence

$$|\eta - \xi| \geq |x - y| - |x - \xi| - |y - \eta| \geq |x - y|/3$$

and thus  $|f(\eta) - f(\xi)| \leq 5[\varepsilon^{-N}]|\eta - \xi|$ . For the Fermat topology one proceeds similarly, using a suitable  $r \in \mathbb{R}_{>0}$  instead of  $[\varepsilon^m]$ .  $\square$

**Example 24.** The function  $i(x) := 1$  if  $x \approx 0$  and  $i(x) := 0$  otherwise is globally Lipschitz with constant 1 with respect to the  $|\cdot|_e$  norm, but it is not locally Lipschitz with respect to the Fermat topology in the sense of Def. 18. In fact, if  $x \approx 0$  and  $y \not\approx 0$ , then  $|i(x) - i(y)|_e = 1$  and the pseudo-valuation  $v(x - y) \leq 0$ , otherwise  $y$  would be infinitesimal. Therefore  $|x - y|_e = e^{-v(x - y)} \geq 1 = |i(x) - i(y)|_e$ . On the other hand,  $i$  is locally Lipschitz in the sharp topology in the sense of Def. 18; indeed, any point can be enclosed in an infinitesimal ball, where the function  $i$  is constant. However,  $i$  is not locally Lipschitz in the Fermat topology. Assume that it verifies

$$|i(0) - i(x)| \leq L \cdot |x| \quad \forall x \in B_r(0), \quad (4.1)$$

where  $r \in \mathbb{R}_{>0}$ . It suffices to take as  $x$  any oscillating number with  $|x| \leq r$  but with  $x_{\varepsilon_k} = 0$  for some sequence  $(\varepsilon_k)_k \downarrow 0$  and  $x_{\eta_k} = r/2$  along another sequence to get that  $x \not\approx 0$  but  $L_{\varepsilon_k} \cdot |x_{\varepsilon_k}| = 0$ . Finally, let us note that taking e.g.  $x = \frac{1}{n}$  in (4.1) we necessarily would have that  $L$  is infinite, as our intuition about the function  $i$  would

suggest. We recall that the map  $i$  is smooth in the sense of [1], it is continuous in the sharp topology and its derivative, in the sense of [1], vanishes everywhere.

Unfortunately, a large number of sets in which one is interested are not compact for the sharp or Fermat topology. For a start, no infinite subset  $U \subseteq \mathbb{R}^n$  is compact w.r.t. to the sharp topology, since its relative topology on  $U$  is the discrete topology. But also internal and strongly internal sets are almost never compact, as the following theorem shows. We recall that  $U \subseteq \tilde{\mathbb{R}}^n$  is closed under finite interleaving if for each  $x, y \in U$  and  $S \subseteq I$  also  $e_S x + e_{S^c} y \in U$ . Any internal set and any strongly internal set is closed under finite interleaving.

**Theorem 25.** *Let  $\tau$  be either the sharp topology or the Fermat topology.*

- (i) *Let  $U \subseteq \tilde{\mathbb{R}}^n$  be closed under finite interleaving. If there exist  $x, y \in U$  with  $x \not\prec_\tau y$ , then  $U$  is not  $\tau$ -compact.*
- (ii) *Let  $K \subseteq \mathbb{R}^n$  and let  $U \subseteq K^\bullet$  with  $K \subseteq \{x^\circ : x \in U\}$ . Then  $U$  is compact in the Fermat topology on  $\tilde{\mathbb{R}}^n$ .*

*Proof.* (i): if  $x \neq y$ , then there exists  $S \subseteq I$  with  $e_S \neq 0$  and  $m \in \mathbb{N}$  such that  $|x - y|_{e_S} \geq [\varepsilon]^m e_S$ . We can find (e.g. by extracting subsequences from  $S$ ) mutually disjoint  $S_n \subseteq (0, 1]$  such that  $S = \bigcup_{n \in \mathbb{N}} S_n$  and  $e_{S_n} \neq 0$  for each  $n$ . Call  $z_n := x e_{S_n} + y e_{S_n^c} \in U$ . If  $p \neq n$ , then

$$|z_p - z_n|_{e_{S_n}} = |y - x|_{e_{S_n}} \geq [\varepsilon]^m e_{S_n}.$$

We show that the sharply open cover  $\{B_{[\varepsilon]^m/3}(x) : x \in U\}$  of  $U$  has no finite subcover. For, suppose it had, then by the pigeon hole principle there would exist  $n \neq p$  such that  $z_n$  and  $z_p$  belong to the same ball  $B_{[\varepsilon]^m/3}(x)$ , whence  $|z_p - z_n|_{e_{S_n}} \leq \frac{2}{3}[\varepsilon]^m e_{S_n}$ , a contradiction. For the Fermat topology one proceeds similarly, using a cover  $\{B_r(x) : x \in U\}$  with a suitable  $r \in \mathbb{R}_{>0}$  instead.

(ii): let  $(A_j)_{j \in J}$  be a cover of  $U$  by large open sets. Then for all  $x \in U$  we can find  $r_x \in \mathbb{R}_{>0}$  and  $j_x \in J$  such that  $B_{r_x}(x^\circ) \subseteq A_{j_x}$ . Therefore, the Euclidean balls  $(B_{r_x}^E(x^\circ))_{x \in U}$  cover  $K$  and we extract a finite subcover  $B_{r_{x_1}}^E(x_1^\circ), \dots, B_{r_{x_n}}^E(x_n^\circ)$ . Hence the balls with the same radius but taken with respect to the generalized absolute value  $B_{r_{x_1}}(x_1^\circ), \dots, B_{r_{x_n}}(x_n^\circ)$  and the corresponding  $A_{j_{x_1}}, \dots, A_{j_{x_n}}$  cover  $K^\bullet$  (and thereby  $U$ ).  $\square$

In the next section, we will show that these restrictions can be overcome if one restricts to certain maps  $f$  with ‘internal structure’.

## 5. THE COLOMBEAU ALGEBRA ON A SUBSET OF $\tilde{\mathbb{R}}^d$

In this section we shall introduce a set of maps which are locally Lipschitz in the sharp topology and includes CGF. We will first introduce the notion of a net  $(u_\varepsilon)$  defining a generalized smooth map  $X \rightarrow Y$ , where  $X \subseteq \tilde{\mathbb{R}}^n$ ,  $Y \subseteq \tilde{\mathbb{R}}^d$ . This is a net of smooth functions  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$  which induces well defined maps of the form  $[\partial^\alpha u_\varepsilon(-)] : \langle \Omega_\varepsilon \rangle \rightarrow \mathbb{R}^d$ , for every multi-index  $\alpha$ .

**Definition 26.** Let  $X \subseteq \tilde{\mathbb{R}}^n$  and  $Y \subseteq \tilde{\mathbb{R}}^d$  be subsets of generalized points. Let  $(\Omega_\varepsilon)$  be a net of open sets of  $\mathbb{R}^n$ , and  $(u_\varepsilon)$  be a net of smooth functions, with  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ . Then we say that

$$(u_\varepsilon) \text{ defines a generalized smooth map } X \rightarrow Y$$

if:

- (i)  $X \subseteq \langle \Omega_\varepsilon \rangle$  and  $[u_\varepsilon(x_\varepsilon)] \in Y$  for all  $x = [x_\varepsilon] \in X$
- (ii)  $\forall [x_\varepsilon] \in X \forall \alpha \in \mathbb{N}^n : (\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}_M^d$ .

The notation

$$\forall [x_\varepsilon] \in X : \mathcal{P}\{(x_\varepsilon)\}$$

means

$$\forall (x_\varepsilon) \in (\Omega_\varepsilon)_M : [x_\varepsilon] \in X \implies \mathcal{P}\{(x_\varepsilon)\}$$

i.e. for all representatives  $(x_\varepsilon)$  of the point  $[x_\varepsilon] \in X$  the property  $\mathcal{P}\{(x_\varepsilon)\}$  holds.

A generalized smooth map is simply a function of the form  $f = [u_\varepsilon(-)]|_X$ :

**Definition 27.** Let  $X \subseteq \tilde{\mathbb{R}}^n$  and  $Y \subseteq \tilde{\mathbb{R}}^d$ , then we say that

$f : X \longrightarrow Y$  is a *generalized smooth function* (GSF)

if there exists a net  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$  defining  $f$  in the sense of Def. 26, such that  $f$  is the map

$$f = [u_\varepsilon(-)]|_X : [x_\varepsilon] \in X \mapsto [u_\varepsilon(x_\varepsilon)] \in Y. \quad (5.1)$$

We will also say that  $f$  is *generated* (or *defined*) *by the net of smooth functions*  $(u_\varepsilon)$ . The set of all GSF  $X \rightarrow Y$  will be denoted by  $\tilde{\mathcal{G}}(X, Y)$ .

Let us note explicitly that definitions 26 and 27 in fact state minimal logical conditions to obtain a set-theoretical map defined by a net of smooth functions. In particular, Proposition 30 below will show that the equality (5.1) is meaningful, i.e. that we have independence from the representatives for all derivatives  $[x_\varepsilon] \in X \mapsto [\partial^\alpha u_\varepsilon(x_\varepsilon)] \in \tilde{\mathbb{R}}^d$ ,  $\alpha \in \mathbb{N}^n$ .

We first show that we can always find globally defined representatives. The generalization where the domains of representatives  $u_\varepsilon$  depend on  $\varepsilon$  (see [27] for a recent survey concerning applications of this generalization) can thus be avoided since it does not lead to a larger class of generalized functions. We will use it also to compare CGF and GSF.

**Lemma 28.** Let  $X \subseteq \tilde{\mathbb{R}}^n$  and  $Y \subseteq \tilde{\mathbb{R}}^d$ . Then  $f : X \longrightarrow Y$  is a GSF if and only if there exists a net  $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$  defining a generalized smooth map  $X \longrightarrow Y$  such that  $f = [v_\varepsilon(-)]|_X$ .

*Proof.* The stated condition is clearly sufficient. Conversely, assume that  $f : X \longrightarrow Y$  is defined by the net  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ . For every  $\varepsilon \in I$  let  $\Omega'_\varepsilon := \left\{ x \in \Omega_\varepsilon \mid d(x, \Omega_\varepsilon^c) > e^{-\frac{1}{\varepsilon}} \right\}$  and choose  $\chi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$  with  $\text{supp}(\chi_\varepsilon) \subseteq \Omega'_{\varepsilon/2}$  and  $\chi_\varepsilon = 1$  in a neighborhood of  $\Omega'_\varepsilon$ . Set  $v_\varepsilon := \chi_\varepsilon \cdot u_\varepsilon$ , so that  $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ . If  $x = [x_\varepsilon] \in X \subseteq \langle \Omega_\varepsilon \rangle$ , then  $x_\varepsilon \in \Omega'_\varepsilon$  for small  $\varepsilon$  by Th. 8, so for all  $\alpha \in \mathbb{N}^n$  we get  $\partial^\alpha v_\varepsilon(x_\varepsilon) = \partial^\alpha u_\varepsilon(x_\varepsilon)$  for small  $\varepsilon$ . Therefore,  $(v_\varepsilon)_\varepsilon$  defines a GSF  $X \longrightarrow Y$  and clearly  $f = [u_\varepsilon(-)]|_X = [v_\varepsilon(-)]|_X$ .  $\square$

We also need to prove that for GSF certain moderateness conditions hold:

**Lemma 29.** Let  $(A_n)_{n \in \mathbb{N}}$  be a decreasing sequence of non-empty, internal, sharply bounded subsets of  $\tilde{\mathbb{R}}^d$ . Let  $(u_\varepsilon)$  be a net of maps  $\mathbb{R}^d \rightarrow \mathbb{R}^m$ . Then for any sharply bounded representatives  $(A_{n,\varepsilon})_\varepsilon$  of  $A_n$ ,

- (i) For all  $[x_\varepsilon] \in \bigcap_{n \in \mathbb{N}} A_n$  we have  $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathbb{R}_M^m$  if and only if  $\exists N \in \mathbb{N} \forall \varepsilon : \sup_{x \in A_{N,\varepsilon} + \varepsilon^N} |u_\varepsilon(x)| \leq \varepsilon^{-N}$ .

- (ii) For all  $[x_\varepsilon] \in \bigcap_{n \in \mathbb{N}} A_n$  we have  $(u_\varepsilon(x_\varepsilon)) \sim 0$  if and only if  $\forall m \in \mathbb{N} \exists N \in \mathbb{N} \forall \varepsilon : \sup_{x \in A_\varepsilon + \varepsilon^N} |u_\varepsilon(x)| \leq \varepsilon^m$ .

*Proof.* (i)  $\Rightarrow$ : By [21, Prop. 2.9], for each  $m \in \mathbb{N}$ , there exists  $\eta_m \in I$  such that for each  $\varepsilon \leq \eta_m$  and  $x \in A_{m,\varepsilon}$ ,  $d(x, A_{k,\varepsilon}) \leq \varepsilon^m$ , for each  $k \leq m$ . W.l.o.g.,  $(\eta_m)_{m \in \mathbb{N}}$  decreasingly tends to 0. By contraposition, let

$$\forall n \in \mathbb{N} \forall \eta \in I \exists \varepsilon \leq \eta : \sup_{x \in A_{n,\varepsilon} + \varepsilon^n} |u_\varepsilon(x)| > \varepsilon^{-n}.$$

Then we can find a strictly decreasing sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  and  $x_{\varepsilon_n} \in A_{n,\varepsilon_n} + \varepsilon_n^n$  such that  $\varepsilon_n \leq \eta_n$  and  $|u_{\varepsilon_n}(x_{\varepsilon_n})| > \varepsilon_n^{-n}$ ,  $\forall n \in \mathbb{N}$ . Choose  $x_\varepsilon \in A_{m,\varepsilon}$ , if  $\eta_{m+1} < \varepsilon \leq \eta_m$  and  $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$ . Then for each  $n \in \mathbb{N}$ ,  $(d(x_\varepsilon, A_{n,\varepsilon}))_\varepsilon \sim 0$ . By [21, Prop. 2.1],  $\tilde{x} := [x_\varepsilon] \in \bigcap_{n \in \mathbb{N}} A_n$  ( $(x_\varepsilon)_\varepsilon$  is moderate, since  $(A_{n,\varepsilon})_\varepsilon$  are sharply bounded). Yet  $(u_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathbb{R}_M^n$ .

(i)  $\Leftarrow$ : let  $[x_\varepsilon] \in \bigcap_{n \in \mathbb{N}} A_n$ . Let  $N \in \mathbb{N}$  as in the statement. By [21, Prop. 2.1],  $(d(x_\varepsilon, A_{n,\varepsilon}))_\varepsilon \sim 0$  for each  $n \in \mathbb{N}$ . In particular,  $x_\varepsilon \in A_{N,\varepsilon} + \varepsilon^N$  for small  $\varepsilon$ . Hence  $|u_\varepsilon(x_\varepsilon)| \leq \sup_{x \in A_{N,\varepsilon} + \varepsilon^N} |u_\varepsilon(x)| \leq \varepsilon^{-N}$  for small  $\varepsilon$ .

(ii) Similar.  $\square$

**Theorem 30.** Let  $X \subseteq \tilde{\mathbb{R}}^n$ . If  $(u_\varepsilon)$  defines a generalized smooth map  $X \rightarrow \tilde{\mathbb{R}}^d$ , then  $\forall [x_\varepsilon], [x'_\varepsilon] \in X : [x_\varepsilon] = [x'_\varepsilon] \Rightarrow (u_\varepsilon(x_\varepsilon)) \sim (u_\varepsilon(x'_\varepsilon))$ .

*Proof.* W.l.o.g.  $u_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$  by Lemma 28. Let  $[x_\varepsilon] \in X$ . Applying Lemma 29 to  $A_n = \{[x_\varepsilon]\}$  and to  $\partial_j u_\varepsilon$  ( $j = 1, \dots, n$ ), we find  $N \in \mathbb{N}$  such that for each  $(x'_\varepsilon) \in \mathbb{R}_M^n$  with  $|x_\varepsilon - x'_\varepsilon| \leq \varepsilon^N$  for small  $\varepsilon$ ,

$$|u_\varepsilon(x'_\varepsilon) - u_\varepsilon(x_\varepsilon)| \leq |x'_\varepsilon - x_\varepsilon| \sup_{|x - x_\varepsilon| \leq \varepsilon^N} |\nabla u_\varepsilon(x)| \leq \varepsilon^{-N} |x'_\varepsilon - x_\varepsilon|, \quad \text{for small } \varepsilon. \quad (5.2)$$

Choosing in particular  $[x'_\varepsilon] = [x_\varepsilon] \in X$ , then  $(x'_\varepsilon) \sim (x_\varepsilon)$ , hence also  $(u_\varepsilon(x_\varepsilon)) \sim (u_\varepsilon(x'_\varepsilon))$  by (5.2).  $\square$

Consequently, the GSF  $f = [u_\varepsilon(-)] : X \rightarrow Y$  is well-defined by its representative. We now turn to the derivatives.

**Theorem 31.** Let  $X \subseteq \tilde{\mathbb{R}}^n$ . If  $f = [u_\varepsilon(-)] \in \tilde{\mathcal{G}}(X, \tilde{\mathbb{R}}^d)$ , then

- (i)  $f : X \rightarrow \tilde{\mathbb{R}}^d$  is locally Lipschitz in the sharp topology
- (ii) If  $A \subseteq X$ ,  $A$  internal, sharply bounded and convex, then  $f : A \rightarrow \tilde{\mathbb{R}}^d$  is Lipschitz
- (iii) If  $x \in \text{int}_s(X)$  and  $f(y) = 0$ , for each  $y$  in a sharp neighborhood of  $x$ , then  $\partial^\alpha u_\varepsilon(x) \sim 0$ ,  $\forall \alpha \in \mathbb{N}^n$ .
- (iv) If  $X$  is sharply open and  $f = [v_\varepsilon(-)]|_X$ , then for each  $\alpha \in \mathbb{N}^n$ ,  $[\partial^\alpha u_\varepsilon(-)] = [\partial^\alpha v_\varepsilon(-)]$ .

*Proof.* (i): by the inequality (5.2),  $f$  is Lipschitz on  $B_{[\varepsilon^N]}([x_\varepsilon])$  for some  $N$ .

(ii): similar to Thm. 30, now applying Lemma 29 to  $A_n = [A_\varepsilon]$  with  $A_\varepsilon$  convex (by Lemma 17).

(iii): let  $N \in \mathbb{N}$  such that  $y \in X$  and  $f(y) = 0$ , for each  $y \in \tilde{\mathbb{R}}^d$  with  $|y - x| \leq \varepsilon^N$ . Let  $x = [x_\varepsilon]$ . We have that  $(\sup_{|x - x_\varepsilon| \leq \varepsilon^N} |u_\varepsilon(x)|)_\varepsilon \sim 0$ , since otherwise, one constructs  $y \in X$  with  $|y - x| \leq \varepsilon^N$  and  $f(y) \neq 0$ . Again by Lemma 29, we find for each  $\alpha \in \mathbb{N}^d$  some  $N \in \mathbb{N}$  such that  $\sup_{|x - x_\varepsilon| \leq \varepsilon^N} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}$ , for small  $\varepsilon$ . The statement now follows similar to [16, Thm. 1.2.3].

(iv): apply (iii) to  $[u_\varepsilon - v_\varepsilon]$ .  $\square$

Consequently, the partial derivative  $\partial^\alpha f := [\partial^\alpha u_\varepsilon(-)]$  on a sharply open  $X \subseteq \widetilde{\mathbb{R}}^n$  is itself a well-defined GSF, and thus satisfies itself the Lipschitz conditions from the previous theorem. We can now show the relationship between GSF and the discontinuous Colombeau differential calculus developed in [1]:

**Proposition 32.** *Let  $a, b \in \mathbb{R}$  with  $a < b$  and  $\widetilde{(a, b)} \subseteq U \subseteq \widetilde{\mathbb{R}}$ , where  $U$  is open in the sharp topology. Let  $f : U \rightarrow \widetilde{\mathbb{R}}$  be a generalized smooth map, then at every point  $x \in \widetilde{(a, b)}$  the function  $f$  is differentiable in the sense of [1] with derivative  $f'(x)$ .*

*Proof.* Let  $f$  be defined by the net of smooth functions  $(u_\varepsilon)$ . Since  $\widetilde{(a, b)} \subseteq U \subseteq \langle \Omega_\varepsilon \rangle$ , Prop. 11 yields  $(a, b) \subseteq \Omega_\varepsilon$  for  $\varepsilon$  small. For all these  $\varepsilon$  and for  $y \in \widetilde{(a, b)}$ , applying to  $u_\varepsilon$  the second order Taylor formula, we get

$$\begin{aligned} |f(y) - f(x) - f'(x) \cdot (y - x)|_e &= |[u_\varepsilon(y_\varepsilon) - u_\varepsilon(x_\varepsilon) - u'_\varepsilon(x_\varepsilon) \cdot (y_\varepsilon - x_\varepsilon)]|_e \\ &= \left| \left[ \frac{1}{2} u''_\varepsilon(\zeta_\varepsilon) \cdot (y_\varepsilon - x_\varepsilon)^2 \right] \right|_e \\ &\leq |[u''_\varepsilon(\zeta_\varepsilon)]|_e \cdot |(y_\varepsilon - x_\varepsilon)|_e^2, \end{aligned}$$

where  $\zeta_\varepsilon \in [x_\varepsilon, y_\varepsilon] \cup [y_\varepsilon, x_\varepsilon] \subseteq (a, b) \subseteq \Omega_\varepsilon$ . The moderateness of  $[u''_\varepsilon(\zeta_\varepsilon)] \in \widetilde{\mathbb{R}}$  (condition (ii) of Def. 26) at  $[\zeta_\varepsilon] \in \widetilde{(a, b)} \subseteq U \subseteq \langle \Omega_\varepsilon \rangle$  yields  $|[u''_\varepsilon(\zeta_\varepsilon)]|_e \leq K$  for some  $K \in \mathbb{R}$ . Thus

$$\lim_{y \rightarrow x} \frac{|f(y) - f(x) - f'(x) \cdot (y - x)|_e}{|(y - x)|_e} = 0,$$

as claimed.  $\square$

**Definition 33.** Let  $X \subseteq \widetilde{\mathbb{R}}^d$ . We call

$$\begin{aligned} \mathcal{E}_M(X, \widetilde{\mathbb{R}}^n) &= \{(u_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^n)^I \mid \forall \alpha \in \mathbb{N}^d \forall [x_\varepsilon] \in X : (\partial^\alpha u_\varepsilon(x_\varepsilon)) \in \mathbb{R}_M^n\}, \\ \mathcal{N}(X, \widetilde{\mathbb{R}}^n) &= \{(u_\varepsilon) \in \mathcal{E}_M(X, \widetilde{\mathbb{R}}^n) \mid \forall [x_\varepsilon] \in X : (u_\varepsilon(x_\varepsilon)) \sim 0\}. \end{aligned}$$

Since  $(u_\varepsilon)$  defines a smooth map  $X \rightarrow \widetilde{\mathbb{R}}^n$  iff  $(u_\varepsilon) \in \mathcal{E}_M(X, \widetilde{\mathbb{R}}^n)$  and  $[u_\varepsilon(-)] = [v_\varepsilon(-)]$  iff  $(u_\varepsilon - v_\varepsilon) \in \mathcal{N}(X, \widetilde{\mathbb{R}}^n)$ , we can identify  $\widetilde{\mathcal{G}}(X, \widetilde{\mathbb{R}}^n)$  with  $\mathcal{E}_M(X, \widetilde{\mathbb{R}}^n)/\mathcal{N}(X, \widetilde{\mathbb{R}}^n)$ .

As a result of Thm. 31, we can also write

$$\mathcal{N}(X, \widetilde{\mathbb{R}}^n) = \{(u_\varepsilon) \in \mathcal{E}_M(X, \widetilde{\mathbb{R}}^n) \mid \forall \alpha \in \mathbb{N}^d \forall [x_\varepsilon] \in X : (\partial^\alpha u_\varepsilon(x_\varepsilon)) \sim 0\}$$

if  $X$  is sharply open.

*Remark 34.*

- (i) In general, if the GSF  $f : X \rightarrow Y$  is defined by the net  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ , the function  $f$  may not be extensible to the whole of  $\langle \Omega_\varepsilon \rangle \supseteq X$  because some derivative  $(\partial^\alpha u_\varepsilon(-))$  can grow stronger than polynomially on  $\langle \Omega_\varepsilon \rangle \setminus X$ . A simple example is given by  $u(x) := e^x$  even for domains  $\Omega_\varepsilon$  such that  $X = \widetilde{\mathbb{R}}_c \subseteq \langle \Omega_\varepsilon \rangle$ . In fact, Th. 11 yields the existence of a sequence  $(\varepsilon_n)_n \downarrow 0$  such that  $[n-1, n+1] \subseteq \Omega_\varepsilon$  for  $\varepsilon \in (0, \varepsilon_n]$ . Therefore, the point  $x$  defined by  $x_\varepsilon := n$  for  $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$  lies in  $\langle \Omega_\varepsilon \rangle \setminus \widetilde{\mathbb{R}}_c$ , but  $u(x_\varepsilon) = (e^{x_\varepsilon}) \notin \widetilde{\mathbb{R}}_M$ . This is a necessary limitation of this approach to generalized functions: indeed, it is not difficult to prove that the only ordered quotient ring where

infinitesimals and order are accessible (i.e. defined similarly to  $\tilde{\mathbb{R}}$ , see [14]) and where every smooth operation is possible, is necessarily the Schmieden-Laugwitz one ([26, 10, 22, 30]).

- (ii) Let  $K \subseteq \mathbb{R}^n$  be a compact set such that  $\tilde{K} \subseteq X \subseteq \tilde{\mathbb{R}}^n$ . Then the GSF  $f : X \rightarrow Y$  is uniquely determined on  $\tilde{K}$  by its values on near standard points (see Sec. 2), i.e.  $f = 0$  on  $\tilde{K}$  iff  $f(x) = 0$  for all  $x \in K^\bullet$ . In fact, suppose that  $f$  vanishes on  $K^\bullet$  but that  $f(x) \neq 0$  for some  $x \in \tilde{K}$ . Then there exist  $m \in \mathbb{N}$  and  $(\varepsilon_k)_k \downarrow 0$  such that  $|u_{\varepsilon_k}(x_{\varepsilon_k})| > \varepsilon_k^m$ , where  $(u_\varepsilon)$  is a net that defines  $f$ . Since  $(x_{\varepsilon_k})_k$  is a sequence in the compact set  $K$ , we can extract a subsequence  $(x_{\varepsilon_{k_l}})_l$  which converges to  $\bar{x} \in K$ . Set  $x'_\varepsilon := x_{\varepsilon_{k_l}}$  if  $\varepsilon \in (\varepsilon_{k_{l+1}}, \varepsilon_{k_l}]$ , then  $x' = [x'_\varepsilon] \in K^\bullet$  since  $\lim_{\varepsilon \rightarrow 0^+} x'_\varepsilon = \lim_{l \rightarrow +\infty} x_{\varepsilon_{k_l}} = \bar{x} \in K$ , but  $f(x') \neq 0$ , a contradiction. This generalizes the analogous property of CGF proved in [17].

Our next aim is to clarify the relation between CGF and GSF.

**Theorem 35.** *Let  $\emptyset \neq A \subseteq \tilde{\mathbb{R}}^d$  be internal and sharply bounded. Then for each sharply bounded representative  $(A_\varepsilon)$  of  $A$ ,*

$$\mathcal{E}_M(A, \tilde{\mathbb{R}}^n) = \{(u_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^n)^I \mid \forall \alpha \in \mathbb{N}^d \exists N \in \mathbb{N} : \sup_{x \in A_\varepsilon + \varepsilon^N} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}, \text{ for small } \varepsilon\}.$$

$$\mathcal{N}(A, \tilde{\mathbb{R}}^n) = \{(u_\varepsilon) \in \mathcal{E}_M(A, \tilde{\mathbb{R}}^n) \mid (\sup_{x \in A_\varepsilon} |u_\varepsilon(x)|) \sim 0\}.$$

*Proof.* The characterization of  $\mathcal{E}_M(A, \tilde{\mathbb{R}}^n)$  follows immediately by the first part of Lemma 29. By the second part of Lemma 29, it follows that  $(\sup_{x \in A_\varepsilon} |u_\varepsilon(x)|) \sim 0$  for each  $(u_\varepsilon) \in \mathcal{N}(A, \tilde{\mathbb{R}}^n)$ . For the converse inclusion, let  $\tilde{x} \in A$ . Then  $\tilde{x} = [a_\varepsilon]$ , with  $a_\varepsilon \in A_\varepsilon$  for small  $\varepsilon$ . By hypothesis,  $(u_\varepsilon(a_\varepsilon)) \sim 0$ . By Theorem 30,  $(u_\varepsilon(x_\varepsilon)) \sim 0$  for any representative  $[x_\varepsilon]$  of  $\tilde{x}$ .  $\square$

We have a similar characterization for more general domains:

**Corollary 36.** *Let  $A = \bigcup_{\lambda \in \Lambda} B_\lambda \subseteq \tilde{\mathbb{R}}^d$ , where each  $B_\lambda$  is nonempty, internal and sharply bounded. Let  $(B_{\lambda, \varepsilon})_\varepsilon$  be a sharply bounded representative of  $B_\lambda$ , for each  $\lambda$ . Then*

$$\mathcal{E}_M(A, \tilde{\mathbb{R}}^n) = \{(u_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^n)^I \mid \forall \alpha \in \mathbb{N}^d \forall \lambda \in \Lambda \exists N \in \mathbb{N} : \sup_{x \in B_{\lambda, \varepsilon} + \varepsilon^N} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}, \text{ for small } \varepsilon\}.$$

$$\mathcal{N}(A, \tilde{\mathbb{R}}^n) = \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(A, \tilde{\mathbb{R}}^n) \mid \forall \lambda \in \Lambda : (\sup_{x \in B_{\lambda, \varepsilon}} |u_\varepsilon(x)|) \sim 0\}.$$

*Proof.* This follows by Theorem 35 because, by definition,  $\mathcal{E}_M(\bigcup_{\lambda \in \Lambda} B_\lambda, \tilde{\mathbb{R}}^n) = \bigcap_{\lambda \in \Lambda} \mathcal{E}_M(B_\lambda, \tilde{\mathbb{R}}^n)$  and  $\mathcal{N}(\bigcup_{\lambda \in \Lambda} B_\lambda, \tilde{\mathbb{R}}^n) = \bigcap_{\lambda \in \Lambda} \mathcal{N}(B_\lambda, \tilde{\mathbb{R}}^n)$ .  $\square$

**Theorem 37.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . Then  $\mathcal{G}(\Omega) = \tilde{\mathcal{G}}(\tilde{\Omega}_c)$ .*

*Proof.* Any  $u \in \mathcal{G}(\Omega)$  has a representative  $(u_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^d)^I$  by the cut-off procedure in Lemma 28. Since  $\tilde{\Omega}_c = \bigcup_{K \subseteq \Omega} \tilde{K}$ , the result follows by Corollary 36.  $\square$

Similarly, since  $\tilde{\mathbb{R}}^d = \bigcup_{n \in \mathbb{N}} \{x \in \tilde{\mathbb{R}}^d : |x| \leq [\varepsilon]^{-n}\}$ ,  $\tilde{\mathcal{G}}(\tilde{\mathbb{R}}^d)$  coincides with the definition of  $\mathcal{G}(\tilde{\mathbb{R}}^d)$  given in [28], where it is also shown that  $\mathcal{G}_\tau(\mathbb{R}^d) \subseteq \tilde{\mathcal{G}}(\tilde{\mathbb{R}}^d)$ .

*Remark 38.* Thus essentially, GSF have a greater flexibility in their domains compared with CGF, which always have a domain of the form  $\tilde{\Omega}_c$ .

- (i) The possibility to define a GSF using a net  $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ , permits to obtain GSF which are defined on purely infinitesimal sets, e.g. starting from  $\Omega_\varepsilon = (-\varepsilon^q, \varepsilon^q)$ , so that we can take  $X = \langle \Omega_\varepsilon \rangle \subseteq B_{[\varepsilon^q]}(0) \subseteq \tilde{\mathbb{R}}$ .
- (ii) Vice versa, we can define GSF on unbounded sets of generalized points. A simple case is the exponential map

$$e^{(-)} : x \in \left\{ x \in \tilde{\mathbb{R}} \mid \exists z \in \tilde{\mathbb{R}}_{>0}^* : |x| \leq |\log z| \right\} \mapsto e^x \in \tilde{\mathbb{R}}.$$

The domain of this GSF cannot be of the form  $\tilde{\Omega}_c$  (which contains only finite points). Analogously, the domain of the map

$$e^{\frac{1}{(-)}} : \left\{ x \in \tilde{\mathbb{R}} \mid \exists z \in \tilde{\mathbb{R}}_{>0}^* : |x^{-1}| \leq |\log z| \right\} \mapsto e^{\frac{1}{x}} \in \tilde{\mathbb{R}}$$

contains a set of infinitesimals that is not of the form  $\tilde{\Omega}_c$ .

Contrary to the case of distributions and CGF, there is no problem in considering the composition of two GSF:

**Theorem 39.** *Subsets  $S \subseteq \tilde{\mathbb{R}}^s$  with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category  $\tilde{\mathcal{G}}$ , the category of GSF.*

*Proof.* By Theorem 31.(i) we already know that every GSF is continuous; we have hence to prove that these arrows are closed with respect to identity and composition in order to prove that we have a concrete subcategory of topological spaces and continuous maps.

If  $T \subseteq \tilde{\mathbb{R}}^t$  is an object, then  $u_\varepsilon(x) := x$  is the net of smooth functions that globally defines the identity  $1_T$  on  $T$ . It is immediate that  $1_T$  is generalized smooth.

To prove that arrows of  $\tilde{\mathcal{G}}$  are closed with respect to composition, let  $S \subseteq \tilde{\mathbb{R}}^s, T \subseteq \tilde{\mathbb{R}}^t, R \subseteq \tilde{\mathbb{R}}^r$  and  $f = [u_\varepsilon(-)] : S \rightarrow T, g = [v_\varepsilon(-)] : T \rightarrow R$  be generalized smooth maps, where we can choose  $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^s, \mathbb{R}^t)$  and  $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^t, \mathbb{R}^r)$  by Lemma 28. Then  $v_\varepsilon \circ u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^s, \mathbb{R}^r)$ . We show that  $(v_\varepsilon \circ u_\varepsilon)_\varepsilon$  defines the GSF  $v \circ u : S \rightarrow R$ . For every  $x = [x_\varepsilon] \in S$ ,  $f(x) = [u_\varepsilon(x_\varepsilon)] \in T$  and thus  $g(f(x)) = [v_\varepsilon(u_\varepsilon(x_\varepsilon))] \in R$ . Consider any  $\gamma \in \mathbb{N}^s$ . It remains to be shown that  $\partial^\gamma(v_\varepsilon \circ u_\varepsilon)(x_\varepsilon) \in \mathbb{R}_M^r$ . We can write

$$\partial^\gamma(v_\varepsilon \circ u_\varepsilon)(x_\varepsilon) = p \left[ \partial^{\alpha_1} u_\varepsilon(x_\varepsilon), \dots, \partial^{\alpha_A} u_\varepsilon(x_\varepsilon), \partial^{\beta_1} v_\varepsilon(u_\varepsilon(x_\varepsilon)), \dots, \partial^{\beta_B} v_\varepsilon(u_\varepsilon(x_\varepsilon)) \right], \quad (5.3)$$

where  $p$  is a suitable polynomial not depending on  $x_\varepsilon$ . Every term  $\partial^{\alpha_i} u_\varepsilon(x_\varepsilon)$  and  $\partial^{\beta_j} v_\varepsilon(u_\varepsilon(x_\varepsilon))$  is moderate by (ii) of Def. 26. Since moderateness is preserved by polynomial operations, it follows that also  $\partial^\gamma(v_\varepsilon \circ u_\varepsilon)(x_\varepsilon)$  is moderate.  $\square$

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